On Interdependent Failure Resilient Multi-path Routing in Smart Grid Communication Network

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Abstract. This paper introduces six new failure-independent multi-path computation problems in complex networks such as smart grid communication network, each of which comes with unique failure interdependency assumptions. Despite the difference of the formulation of the problems, we show that each of the problems can be reduced to another within polynomial time, and therefore they are equivalent in terms of hardness. Then, we show that they are not only $\mathcal{NP}$-hard, but also cannot be approximated within a certain bound unless $\mathcal{P} = \mathcal{NP}$. Besides, we show that their decision problem versions to determine if there exist two failure independent paths between two given end nodes are still $\mathcal{NP}$-complete. As a result, this paper opens a new series of research problems with daunting complexity based on important real world applications.

1 Introduction

Smart grid is an automated modern power supply network, which collects the real-time knowledge of the electricity producers and consumers, as well as of the status of power delivery infrastructure itself and exploits such knowledge to improve the overall efficiency, reliability, sustainability, and the economics of the production and the distribution of electricity [1]. To achieve the real-time data collection throughout the system, smart grid is equipped with a communication network, which is tightly coupled with its power supply network. It is known that communication reliability, i.e. on-time message delivery, is a highly critical issue of smart grid communication network. In order for communications in a smart grid communication network to be reliable, a message sent from a source to a destination has to be delivered on time. A message in smart grid communication network is delivered from a source to a destination throughout multiple routers as is like the other traditional multi-hop routing networks. In network theory, the process of identifying the best possible message routing path from the source to the destination along with the intermediate routers on the path is called as a routing problem. Usually, the goal of many routing problems in existing networks such as the Internet is to find a path with least latency from a source to a destination. However, a routing problem with the communication reliability in mind is somehow different from this: In case that a message is sent over a single routing path and it is not delivered due to some issues on a router in the path or a communication link between routers in the path, the message transmission will fail. Once the source realizes the failure, it needs to retransmit the message over a different routing path. In a reliability-critical networks such as smart grid communication network, such failure may incur a disastrous consequence. In order to improve the communication reliability, one well-known approach is multi-path routing, i.e. identifying multiple failure-independent paths and sending a copy of a message over each of the paths. Certainly, this approach can be used to improve the communication reliability in smart grid communication networks. To apply this approach, one fundamental problem is *how to identify the maximum number of failure-independent paths in a given smart grid communication network*, which is also the central problem of interest in this paper.

So far, various multi-path routing algorithms have been introduced for reliable communications in the literature [2–10]. At a glance, one may think our problem of interest might be easily solved by using one of the existing multiple routing path based strategies. However, this is not true as a multiple-path problem in smart grid communication network has a salient feature that the failures at a node or at a link may affect other nodes and links, while in case of the references, failures are assumed to be independent from each other [26]. Meanwhile, the recent natural disasters such as the massive Tohoku Earthquake in Japan have demonstrated that such severe natural disaster may fail more than one network router at the same time. Motivated by such events, a number of network reliability related studies have been conduced based on unique failure models [15–17, 19, 18, 20, 21]. However, their main focus is on the survivability analysis of the existing network topology against the natural disasters, especially earthquake. Therefore, these are not applicable to our study. In [22], Zhang and Perrig have studied the problem of selecting k failure-independent paths among a given paths based on the history information. As our work focuses on computing the maximum number of failure independent paths between two nodes in a given graph, their work is too restricted to identify failure-independent paths in a given smart grid communication network.

To the best of our knowledge, the closest work to our problem of interest is the one by Hong et. al. [26], in which they introduced a new multi-path routing path computation problem in smart grid communication network. Based on the recent reports which show that the most common failure in smart grid communication network is node failure [11–13] and that a node failure may affect other nodes [14], the authors defined the notion of failure independency between nodes, i.e. a node v is failure-independent from another node u if the failure of v does not cause the failure of u and vice versa. Based on this notion, they defined two paths P_1 and P_2 are non-disrupting paths if for every node pair u, v such that $u \in P_1$ and $v \in P_2$, u and v are failure-independent from each other. Then, the studied how to find a k multiple non-disrupting paths from a source s to a destination t for a given constant k.

Main Contributions. Throughout our comprehensive literature survey, we have realized that there is an urgent need to study multi-path routing problems with different interdependent failure models for various complex communication networks. At the same time, however, there is a generally lack of such effort so far. To address this issue, we study the **maximum non-disrupting path problem (MNP)** under different failure model. Note that the problems of our interest can be viewed as a dual of the problem studied by Hong et. al. [26] whose goal is to find k failure-independent with a required quality, where k is a given constant. We also remove the following two strong assumptions of Hong et. al.'s work in some of our problem models: (a) failures occurs only at nodes (routers), and (b) the nodes can be partitioned to several node disjoint subsets, in which a node failure in the subset will affect the rest of the nodes in the same subset only, but not the nodes in other subsets. Below is the outline of our new problems:

(a) (General) MNP: this problem aims to find the maximum number of failure-independent paths from a source to a destination under the assumption that (i) both node and edge can fail, (ii) each node/edge is included in one or more subset (edge/node combined), and (iii) a failure of a node/edge in a subset cause the failure of all elements (node/edge) in the same subset.

(b) Node-wise MNP (NMNP): this problem aims to find the maximum number of failure-independent paths from a source to a destination under the assumption that (i) only node can fail, (ii) each node is included in one or more node subsets, and (iii) a failure of a node in a subset cause the failure of all nodes in the same subset.

(c) Edge-wise MNP (EMNP): this problem aims to find the maximum number of failure-independent paths from a source to a destination under the assumption that (i) only edge can fail, (ii) each edge is included in one or more edge subsets, and (iii) a failure of an edge in a subset cause the failure of all edges in the same subset.

(d) Mono-coloring MNP (MMNP): this problem aims to find the maximum number of failure-independent paths from a source to a destination under the assumption that (i) both node and edge can fail, (ii) each node/edge is included in at most one subset (edge/node combined), and (iii) a failure of a node/edge in a subset cause the failure of all elements (node/edge) in the same subset.

(e) Node-wise MMNP (NMMNP): this problem aims to find the maximum number of failure-independent paths from a source to a destination under the assumption that (i) only node can fail, (ii) each node is included in at most one node subset, and (iii) a failure of a node in a subset cause the failure of all nodes in the same subset.

(f) Edge-wise MMNP (EMMNP): this problem aims to find the maximum number of failure-independent paths from a source to a destination under the assumption that (i) only edge can fail, (ii) each edge is included in at most one edge subset, and (iii) a failure of an edge in a subset cause the failure of all edges in the same subset.

Despite the difference in constraints, there exist polynomial-time reduction between any two of above versions. We also prove MNP is $\mathcal{NP}$-hard, which makes all of the variations to be $\mathcal{NP}$-hard. Furthermore, all of them are impossible to approximate with performance ratio $l^{1/2-\epsilon}$, or even $l^{1-\epsilon}$ for some certain versions, where l denotes the size of input colored elements, and $\epsilon \in \mathbb{R}^+$ is any positive number. On the other hand, it is a $\mathcal{NP}$-complete problem for not only their decision problems, but the decision problem with the limitation that whether there exist $k = 2$ disjoint paths ending with fixed nodes s and t in a given connected graph.

Organizations. The rest of this paper is organized as follows. In Section 2, we formulate MNP in mathematical form. Variable versions derived from different restrictions of the general model are given in Section 3. Polynomial-time reductions between versions including the universal problem are given in Section 4. Section 5 shows the complexity of above problems from kinds of aspects.

2 Problem Statement

In this section, we will give a explicit definition of the problem in mathematical form. Actually, there are lots of variations (and we call it versions in the following) for the problem. So we just formulate a general pattern and restrict it into different versions in later statement.

Definition 1 ((Color) Non-disrupting) *Given a graph $G = (V, E)$ with a color mapping $c : H \to 2^{\mathrm{COLOR}}$, where H is a collection of elements in graph G such as the vertex set V or the edge set E or the union of their subsets, and 2^{COLOR} represents all subsets of a color set* COLOR*. We call element collection sets I and J are color non-disrupting (or non-disrupting for brief without confusion) if $I, J \subseteq (V \cup E)$ such that $c(I \cap H) \cap c(J \cap H) = \varnothing$.*

It is important to notice tha in the context of interdependent failure, we define that a failure of an element in a subset results in the total failure of all elements in the subset. More generally, we also call that a group of finite sets are non-disrupting if the number of graph element collection sets is strictly more than 2. Formally definition is as follows.

Definition 2 (Group (Color) Non-disrupting) *Given a graph $G = (V, E)$ with a color mapping $c : H \rightarrow 2^{\text{COLOR}}$, where H is a collection of elements in graph G, and 2^{COLOR} represents all subsets of a color set* COLOR*. I_1, I_2, ..., I_r are a group of graph element collection sets. I_1, I_2, ..., I_r are color non-disrupting if for all $j \neq k \in [r] = \{1, 2, \dots, r\}$, such that $I_j, I_k \subseteq (V \cup E)$, and $c(I_j \cap H) \cap c(I_k \cap H) = \varnothing$.*

A intuitive observation of color non-disruption set of a given colored graph is that none of the graph element collections share the same color. On the other word, any color exists in at most a single element collection. More particularly, one do not care about the color mapping on all sorts of element collection of the graph. In some specially case, extraordinary subgraphs are taken into crucial consideration. For instance, stars take a significant role in centering communication, cliques are used more frequently in the study of society community, complete bipartite subgraph works more properly in direct transformation between two places. In this paper, we focus on the path case which is broadly applying to undirect transformation between two places.

Definition 3 (Non-disrupting Paths) *Given a graph $G = (V, E)$ with a color mapping $c : H \rightarrow 2^{\text{COLOR}}$, where H is a collection of elements in graph G, and 2^{COLOR} represents all subsets of a color set* COLOR*. P_1, P_2, ..., P_r are color non-disrupting paths, if they are color non-disrupting and all of the them are paths.*

In the following statement, we formulate a problem totally based on non-disrupting paths, and it is called maximum non-disrupting paths problem (MNP).

Problem 1 (Maximum Non-disrupting Paths Problem, MNP[26]) *Given a connected graph $G = (V, E)$ with two specified nodes s and t. Let $c : (V \setminus \{s, t\}) \cup E \rightarrow 2^{\text{COLOR}}$ be a color mapping from elements of graph G to a given color set* COLOR*, where 2^{COLOR} represents all subsets of the color set* COLOR*. Find color non-disrupting paths from s to t with maximum cardinality, and denote the number by* $\text{MNP}(G)$.

Note that the we define the color mapping c from $V \setminus \{s, t\} \cup E$ to all subsets of COLOR in maximum non-disrupting paths problem. Actually, the definition implies that it is allowed that some of these elements are uncolored since $\varnothing$ is also a subsets of COLOR. To simplify and clarify the notation explanation, denoted by $c_V : V \setminus \{s, t\} \rightarrow 2^{\text{COLOR}}$ and $c_E : E \rightarrow 2^{\text{COLOR}}$, the restriction of color mapping c on the vertex set $V \setminus \{s, t\}$ and the edge set E respectively. Namely, $c_V = c|_{V \setminus \{s,t\}}$ and $c_E = c|_E$.

3 Variations of MNP

We have formulated a general form of MNP in last section. Next, some variation of MNP for different restriction will be introduction in this section. It is not needed such universal for the coloring mapping in the definition of MNP under some special circumstance. Only nodes color will be considered if one just care about the station factor, and one merely take the coloring mapping on links into consideration if the aim is purely studying the transformation process. More addition, color uniqueness might be demanded for a single graph element in some cases. Namely, more than two colors correspond to a single graph element is not allowed for some particular reasons. Due to above cases, general model is such universal that far more beyond properness sometimes. Some specified problems with respect to MNP are proposed as follows.

Table 1. Kinds of Variations of Maximum Non-disrupting Paths Problem

	General (G)	Mono-coloring (M)
Mixed (MI)	MNP	MMNP
Node (N)	NMNP	NMMNP
Edge (E)	EMNP	EMMNP

Problem 2 (Node-wise MNP, NMNP) *Given a connected graph $G = (V, E)$ with two specified nodes s and t. Let $c_V : V \setminus \{s, t\} \to 2^{\mathrm{COLOR}}$ be a node color mapping from the vertex set to all subsets of color set* COLOR*. Find the maximum number of color non-disrupting paths from s to t.*

Problem 3 (Edge-wise MNP, EMNP) *Given a connected graph $G = (V, E)$ with two specified nodes s and t. Let $c_E : E \to 2^{\mathrm{COLOR}}$ be a edge color mapping from the edge set to all subsets of color set* COLOR*. Find the maximum number of color non-disrupting paths from s to t.*

Problem 2 and Problem 3 restrict the domain of coloring mapping to the node set and the edge set of the graph respectively. According to this limitation, two similar but practical model appear. Further more, as it is previously mentioned, uniqueness of coloring is also a necessary restriction in some situations.

Problem 4 (Mono-coloring MNP, MMNP) *Given a connected graph $G = (V, E)$ with two specified nodes s and t. Let $c : (V \setminus \{s, t\}) \cup E \to \mathrm{COLOR} \cup \{\varnothing\}$ be a color mapping from graph element set $(V \setminus \{s, t\}) \cup E$ to a given color set* COLOR *together with the empty set $\varnothing$. Find color non-disrupting paths from s to t with maximum cardinality.*

Similarly, one can also restrict the coloring mapping strictly to the node set or the edge set no matter whether the coloring is mono-restricted.

Problem 5 (Node-wise MMNP, NMMNP) *Given a connected graph $G = (V, E)$ with two specified nodes s and t. Let $c_V : V \setminus \{s, t\} \to \mathrm{COLOR} \cup \{\varnothing\}$ be a color mapping from the vertex set to a color set $\mathrm{COLOR} \cup \{\varnothing\}$. Find the maximum number of color non-disrupting paths from s to t.*

Problem 6 (Edge-wise MMNP, EMMNP) *Given a connected graph $G = (V, E)$ with two specified nodes s and t. Let $c_E : E \to \mathrm{COLOR} \cup \{\varnothing\}$ be a color mapping from the edge set to a color set $\mathrm{COLOR} \cup \{\varnothing\}$. Find the maximum number of color non-disrupting paths from s to t.*

In summary, there are 6 variations of MNP in total as shown in Table 1. Besides, their corresponding characteristics are shown in Table 2.

4 Polynomial-time Reduction between Variations

Besides the general form of MNP, 5 variations have been proposed in last section. As we can see in this section, these versions, together with the original problem (MNP), can be polynomial-time reduction from any one to another, although all other versions are actually a restriction of the original model.

Lemma 1 (Node-edge Reduction) *There exists polynomial-time reduction from mixed form of MNP to corresponding node version or edge version (e.g., from MNP to EMNP, or from MMNP to NMMNP).*

Table 2. Characteristics for Variations of Maximum Non-disrupting Paths Problem

Variations	Coloring Mapping
MNP	$c : V \setminus \{s, t\} \cup E \to 2^{\text{COLOR}}$
NMNP	$c_V : V \setminus \{s, t\} \to 2^{\text{COLOR}}$
EMNP	$c_E : E \to 2^{\text{COLOR}}$
MMNP	$c : V \setminus \{s, t\} \cup E \to \text{COLOR} \cup \{\varnothing\}$
NMMNP	$c_V : V \setminus \{s, t\} \to \text{COLOR} \cup \{\varnothing\}$
EMMNP	$c_E : E \to \text{COLOR} \cup \{\varnothing\}$

Proof. The lemma state 2 facts that the mixed form of MNP can reduce to either the node version or the edge version in polynomial time. Actually, comparing with converting all mixed form instance to both the node version instance and the edge version instance, we prefer to show a color mapping transformation procedure from node coloring to edge coloring and the other side. In that case, one can complete the reduction freely from arbitrary mixed form instance to a node version or a edge version instance, if he finds all inappropriate coloring and replaces it by proper elements (nodes or edges) coloring. Next, polynomial-time coloring transformation will be shown in the following.

Edge coloring to vertex coloring. Assume the simple connected graph $G = (V, E)$ with two specified nodes s and t, and the color mapping $c : (V \setminus \{s, t\}) \cup E \to 2^{\text{COLOR}}$. This assumption is reasonable since all other versions are just the restriction of this universal case. Also assume that $e \in E$ has an inappropriate edge coloring and following steps help us to convert this edge coloring to equivalent node coloring: Step 1. subdivide e into a 2-path by a vertex v_e, and Step 2. color the vertex v_e by $c(e)$, and remove the edge coloring respect to e. Figure 1 illustrates the procedures of the coloring adjustment.

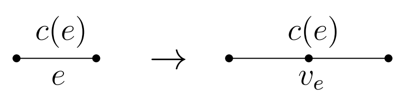


Fig. 1. Illustration of Transfer Procedures form Edge Coloring to Vertex Coloring

Actually, this adjustment remains the colors of all s-t paths. Namely, there is a bijection between paths in initial graph and in adjusted graph such that they share the same colors. Following statement tells the details. Let $e = uw$ and $G' = (V', E') = (V \cup \{v_e\}, E \setminus \{e\} \cup \{uv_e, wv_e\})$ with color mapping

$$c' : (V' \setminus \{s, t\}) \cup E' \to 2^{\text{COLOR}}, x \mapsto \begin{cases} c(x), \ x \in (V \setminus \{s, t\}) \cup E, \\ c(e), \ x = v_e, \\ \varnothing, \quad x \in \{uv_e, wv_e\} \end{cases} .$$

For any s-t path P in G, replace elements respect to the edge $e(\in G)$ by corresponding elements respect to 2-path $uv_ew(\in G')$. Specifically, remove edge elements $u - e - w$ (or $w - e - u$) and add the 2-path elements $u - uv_e - v_e - wv_e - w$ (or $w - wv_e - v_e - uv_e - u$) if e exists in the path $P(\in G)$, and keep the all elements unchanged if e does not appear. These procedures naturally induce a bijection between paths in G and paths in G', and the bijection keeps the colors in paths between G and G'. Hence, if P_1 and P_2 in G are color non-disrupting paths from s to t, then its corresponding paths P'_1 and P'_2 in G' are also, of course, color non-disrupting from s' to t', according to the construction of G'. On the other hand, non-disrupting paths P_1' and P_2' in G' contribute to its corresponding initial paths P_1 and P_2 in G are non-disrupting due to the same reason. Above properties clearly show the equivalence of the transformation and, of course, it finishes in polynomial time.

Vertex coloring to edge coloring. Similarly, assume the simple connected graph $G = (V, E)$ with two specified nodes s and t, and the color mapping $c : (V \setminus \{s,t\}) \cup E \to 2^{\text{COLOR}}$. $v \in V \setminus \{s,t\}$ has been inappropriate colored and following method help us to replace the node coloring by equivalent edge coloring: Step 1. split v into $d_G(v)$ isolate vertices V_v and pend them to all neighbors of v respectively, where $d_G(v)$ represents the degree of vertex v in graph G, Step 2. join V_v into a clique K_v with edge set E_v, and Step 3. color all edges in E_v by $c(v)$, and wipe all colors corresponding to vertex v. Figure 2 shows the steps of vertex coloring to edge coloring.

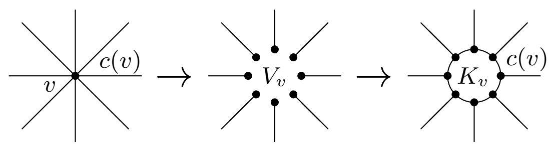


Fig. 2. Illustration of Transfer Procedures form Vertex Coloring to Edge Coloring

One can construct two mappings between all s-t paths in original graph and in created graph. One is from s-t paths in G to s-t paths in adjusted graph, and the other has the opposite direction. Unfortunately, both of them are not one-to-one. Nevertheless, the transformation is still equivalent with this unavoidable disadvantage. Following mathematical form will give a clear explanation of the fact. Let $G' = (V', E') = (V \setminus \{v\} \cup V_v, E \setminus vN_G(v) \cup M(V_v, N_G(v)) \cup E_v)$, where $N_G(v)$ represents all neighbors of vertex v in graph G. Meanwhile $vN_G(v) = \{uv | u \in N_G(v)\}$ denotes edges incident to the vertex v in graph G, and $M(V_v, N_G(v)) = \{$A perfect matching consisting of edges $uw | u \in V_v, w \in N_G(v)\}$ refers to corresponding replaced edges of those edges incident to vertex v. The corresponding color mapping

$$c' : (V' \setminus \{s,t\}) \cup E' \to 2^{\text{COLOR}}, x \mapsto \begin{cases} c(x), & x \in (V \setminus \{s,t\}) \cup E, \\ \varnothing, & x \in V_v, \\ c(uv), & x \in uV_v, u \in N_G(v), \\ c(v), & x \in E_v, \end{cases}.$$

One must be emphasize that $M(V_v, N_G(v)) \subseteq \bigcup_{u \in N_G(v)} uV_v$, which leads to the definition of color mapping c' is feasible. The mapping between s-t paths in G and in G' follows the rule that replacing elements respect to vertex v by corresponding elements respect to the clique K_v, for an arbitrary s-t path P in original graph G. Without lose of generality, assume that $v \in P$, otherwise let path $P' = P$ in graph G' corresponds to path P in graph G. According to assumption, s-t path P can be expressed as $s - \cdots - u - uv - v - vw - w - \cdots - t$, where u, w represent the neighbors of v in P and they might be the source s or the sink t. Remove the part $u - uv - v - vw - w$ and take part $u - uv_u - v_u - v_uv_w - v_w - v_ww - w$ as a replacement, where $u, w \in N_G(v)$ and v_u, v_w denote split vertex in V_v adjacent to u, w respectively (shown in Figure 3). On the other hand, contract vertices in K_v into a vertex and simplify the obtained walk into a path by vertex and edge deletion, when we are proposed to convert a s-t path in G' to a s-t path in G. No element will be changed if all vertices

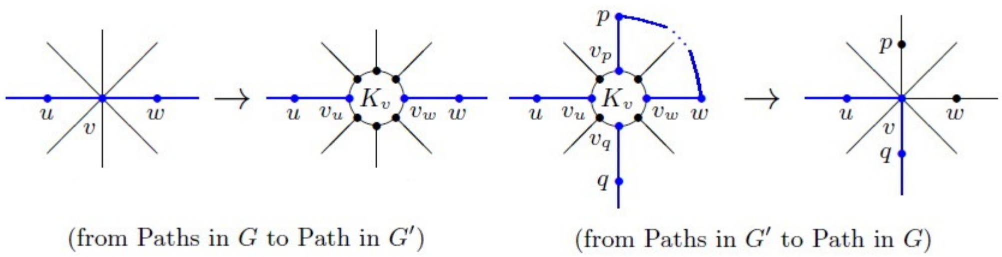


Fig. 3. Illustration of Paths Construction

in V_v are not mentioned in the path. Otherwise, do the reverse replacement and delete all loops and cycles to make it a real path in G. Take path $P: s - \cdots - u - uv_u - v_u - v_uv_w - v_w - v_ww - w - \cdots - p - pv_p - v_p - v_pv_q - v_q - v_qq - q - \cdots - t$ as an example. The first step replace $u - uv_u - v_u - v_uv_w - v_w - v_ww - w$ and $p - pv_p - v_p - v_pv_q - v_q - v_qq - q$ by $u - uv - v - vw - w$ and $p - pv - v - vq - q$ respectively. Next, delete the cycle $v - vw - w - \cdots - p - pv - v$ to acquire a path $P': s - \cdots - u - uv - v - vq - q - \cdots - t$ in graph G'.

According to above path mappings, any two color non-disrupting paths P_1, P_2 in G derive two color non-disrupting paths P_1', P_2' in G', and vise versa. Therefore $\mathrm{MNP}(G) = \mathrm{MNP}(G')$ and the transformation is polynomial-time as well. As it is said previously, one can find all inappropriate coloring (all node coloring or all edge coloring) and convert them into proper coloring by the transformation between vertex coloring and edge coloring. These procedure is polynomial-time since there exists at most $\max\{|V|, |E|\}$ inappropriate coloring and each coloring transformation is in polynomial time. Hence, one can reduction from any mixed MNP instance into an equivalent node version instance or edge version instance in polynomial time. That is to say, there exists polynomial-time reduction from mixed form of MNP to corresponding node version or edge version.

Lemma 2 (Mono-coloring Reduction) *There exists polynomial-time reduction from general form of MNP to corresponding mono-coloring version (from MNP to MMNP, etc.).*

Proof. Without lose of generality, one only need to consider the node case or the edge case according to Lemma 1. We take the node case as an example to give a proof of the lemma in the following. Obviously, mono-coloring version is actually a special case of the general case for no matter node version, edge version and mixed version. Hence, the only task of the proof is the reduction from general form to mono-coloring version. And we will illustrate the lemma by showing a polynomial-time transformation from arbitrary NMNP instance to a NMMNP instance. Similarly as the proof of Lemma 1, we only show the transformation for any single vertex and do the same operator for the others. Assume the simple connected graph $G = (V, E)$ with two specified nodes s and t, and the color mapping $c : (V \setminus \{s, t\}) \cup E \to 2^{\mathrm{COLOR}}$. Vertex $v \in V$ are multi-colored. Following steps give a method to find a equivalent graph with strictly less multi-colored elements: Step 1: split v into $d_G(v)$ isolate vertices V_v and pend them to all neighbors of v respectively, where $d_G(v)$ represents the degree of vertex v in graph G. Step 2: join V_v into a clique K_v with edge set E_v. Step 3: subdivide each edge $e(v) \in E_v$ into a path with vertex set $V_{e(v)}$. Donate the path corresponding to the edge $e(v)$ with length $(|c(v)| + 1)$ by $P_{e(v)}$. Step 4: color vertices in set $V_{e(v)}$ (with cardinal $|c(v)|$) for all $e(v) \in E_v$ by color set $c(v)(\subseteq \mathrm{COLOR})$ according to the principle that each vertex corresponds to a single distinct color. Mathematically, the adjusted graph

$$G' = (V', E') = (V \setminus \{v\} \cup V_v \cup \bigcup_{e(v) \in E_v} V_{e(v)}, E \setminus vN_G(v) \cup M(V_v, N_G(v)) \cup E(P_{e(v)}))$$

where $N_G(v)$ represents all neighbors of vertex v in graph G and $E(P_{e(v)})$ refers to the edges in path $P_{e(v)}$. Meanwhile $vN_G(v) = \{uv | u \in N_G(v)\}$ denotes edges incident to the vertex v in graph G, and

$$M(V_v, N_G(v)) = \{\text{A perfect matching consisting of edges } uw | u \in V_v, w \in N_G(v)\}$$

refers to corresponding replaced edges of those edges incident to vertex v. The corresponding color mapping

$$c' : (V' \setminus \{s,t\}) \cup E' \to 2^{\text{COLOR}}, x \mapsto \begin{cases} c(x), & x \in (V \setminus \{s,t\}) \cup E, \\ \varnothing, & x \in V_v \cup E(P_{e(v)}), \\ \tau(x), & x \in V_{e(v)}, e(v) \in E_v \\ c(uv), & x \in uV_v, u \in N_G(v), \end{cases} .$$

where $\tau : V_{e(v)} \to c(v)$, $x \mapsto \tau(x)$, for any fixed $e(v) \in E_v$ is bijection. In addition,

$$M(V_v, N_G(v)) \subseteq \bigcup_{u \in N_G(v)} uV_v$$

leads to the definition of color mapping c' is feasible. The s-t path mapping between G and G' are similar as it is shown in the proof of Lemma 1. Take the path P in G as an example. The adjustment of P' from P is removing the vertex v as well as its incident edges, and add corresponding elements respect to K_v (including the elements produced in subdivision) in graph G'. On the other hand, assume the s-t path in G' is P'. Deleting all cycles (including loops) after identifying all adding vertices consisting of V_v and $\bigcup_{e(v) \in E_v} V_{e(v)}$ of path P' to obtain a s-t path P in graph G. This path transformation method ensure that color non-disrupting paths $P_1, P_2, \ldots$ in G derive color non-disrupting paths $P'_1, P'_2, \ldots$ in G' with exactly same cardinality. Hence, $\text{MNP}(G) = \text{MNP}(G')$ and the transformation is in polynomial time. Namely, there exists polynomial-time reduction from general form of MNP to corresponding mono-coloring version.

Theorem 3 (MNP Versions Reduction) *There exist polynomial-time reduction from any one in Table 1 to another.*

Proof. Lemma 1 tells the existence of polynomial reduction from one version to another version vertically in Table 1, while Lemma 2 produce a method of polynomial reduction from left side to the right side or from the right side to the left in the same row. To sum up, the conclusion is proved.

5 Complexity of MNP

We have proved that all versions of MNP in Table 1 have polynomial reduction to each other in last section. In addition, all these versions are $\mathcal{NP}$-hard in algorithmic aspect. In fact, only one version need to be considered when proving under the condition that all versions of MNP have polynomial reduction. As it will be seen in the following that NMNP, the general version with color mapping on nodes, is $\mathcal{NP}$-hard, if maximum independent set problem (MIS) is $\mathcal{NP}$-hard. Before the statement of the theorem, some preliminaries are shown.

Definition 4 (Maximum Independent Set Problem, MIS[27]) *In graph theory, an independent set is a set of vertices in a graph, no two of which are adjacent. A maximum independent set is an independent set of largest possible size for a given graph G. This size is called the independence number of G, and denoted by $\alpha(G)$.*

As it is known, MIS is a typical $\mathcal{NP}$-hard problem. Further more, it also has some results of approximation hardness.

Theorem 4 (Approximation Hardness of Independence Number[31]) *Given a graph $G = (V, E)$, it is hard to estimate its independence number $\alpha(G)$ with performance ratio in $n^{1-\epsilon}$ for all $\epsilon > 0$, unless $\mathcal{P} = \mathcal{NP}$.*

Next, we will show the complexity of MNP.

Theorem 5 ($\mathcal{NP}$-hard) *All versions of MNP in Table 1 are $\mathcal{NP}$-hard.*

Proof. As it is said, in order to give a proof the theorem, we only need to prove any version is $\mathcal{NP}$-hard according to Theorem 3. We take NMNP as an instance.

In the rest of the proof, we prefer to give a polynomial reduction from MIS to NMNP. Given any simple graph $G = (V, E)$, one can find a graph $G' = (V', E')$ with special nodes s, t and a color mapping $c_V : V' \setminus \{s, t\} \to \text{COLOR}$, such that the independence number $\alpha(G)$ is exact equal to the cardinality of maximum color non-disrupting paths MNP(G'). In that case, following statement are equivalent for arbitrary positive integer $k \in \mathbb{N}_+$.

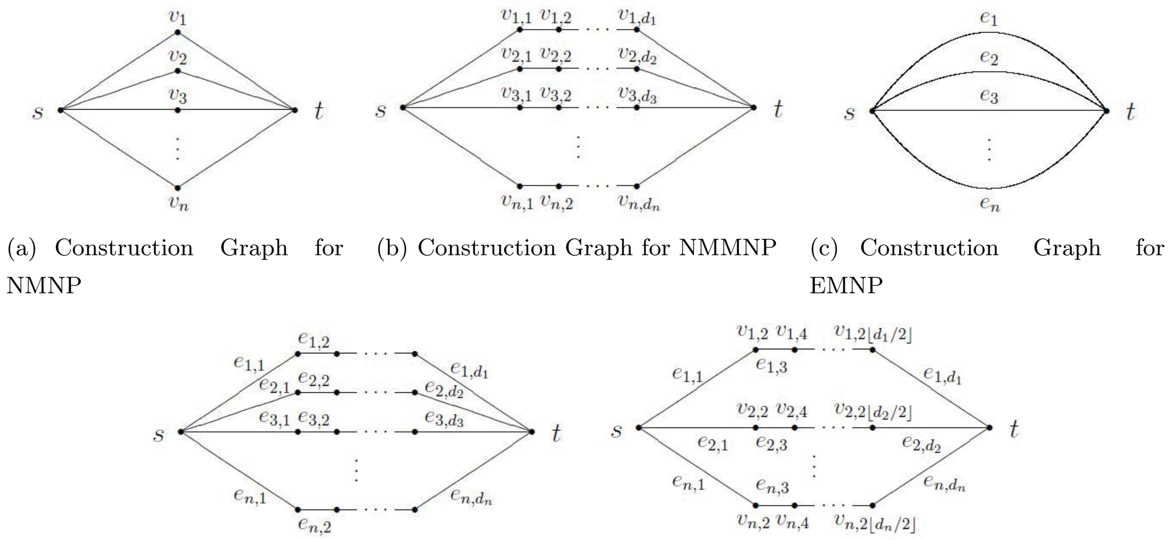
- G has an independent set $I_S(\subseteq V)$ with cardinality at least k.
- G' has a path set P_S with at least k non-disrupting paths in.

Actually, above statement is just the positive answer for the decision problem of corresponding problem. Namely, the answer for the decision problem of MIS when the instance is $I : G$ is yes, if and only if the answer for the decision problem of MNP when the instance is $I' : (G', c_V)$ is yes.

Assume the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$, where $n = |V|$ and $m = |E|$. Let $G' = (V', E') = \Big(\{s, t\} \cup V,\ \bigcup_{i=1}^{n}\{sv_i, v_it\}\Big)$ together with the color mapping $c_V : V \to 2^{[m]}, v_i \mapsto \{k | v_i \text{ is incident to } e_k\ , k \in [m]\}$, where $[m] = \{1, 2, \ldots, m\}$. For above construction, G and G' hold following properties: (a) every candidate vertex v_i in G there exists a corresponding candidate path $P_i := sv_it$ in G' for all $i \in [n]$ and vice versa, (b) vertices v_i and v_j are adjacent in G ($v_iv_j \in E$) if and only if paths $P_i := sv_it$ and $P_j := sv_jt$ share the same color ($c_V(v_i) \cap c_V(v_j) \neq \varnothing$), for all $i \neq j$, $i, j \in [n]$, and (c) there is a bijection between answers for G of MIS and answers for G' of NMNP, which implies $\alpha(G) = \text{MNP}(G')$. Hence a polynomial transformation from graph G to graph G' with color mapping c_V finishes. Namely, there exists a polynomial reduction from MIS, a well-known $\mathcal{NP}$-hard problem, to NMNP, a special version of MNP. The proof is complete because of the existence of Theorem 3.

Theorem 5 tells not only the general case, but all 6 versions of MNP are $\mathcal{NP}$-hard. Actually, it cannot give a exact expression of the complexity for MNP. Following results show that MNP is far more than $\mathcal{NP}$-hardness at the point of algorithmic view.

Theorem 6 (Hardness of Approximation) *MNP is not only a $\mathcal{NP}$-hard problem, but hard to approximate, unless $\mathcal{P} = \mathcal{NP}$. To be exact, it is impossible to approximate with corresponding performance ratio in Table 3 for all versions of MNP, unless $\mathcal{P} = \mathcal{NP}$.*



(a) Construction Graph for NMNP (b) Construction Graph for NMMNP (c) Construction Graph for EMNP

(d) Construction Graph for EMMNP (e) Construction Graph for MMNP

Fig. 4. Approximation solutions of k-TSPN (in Fig. (e)) and k-PCPN (in Fig. (f)) can be obtained from an approximation of k-TCPN (in Fig. (d)).

Table 3. Imapproximation for Versions of MNP

	General (G)	Mono-coloring (M)
Mixed (MI)	$m^{1-\epsilon}$	$(n+m)^{\frac{1}{2}-\epsilon}$
Node (N)	$n^{1-\epsilon}$	$n^{\frac{1}{2}-\epsilon}$
Edge (E)	$m^{1-\epsilon}$	$m^{\frac{1}{2}-\epsilon}$

Note that $n = |V|$ and $m = |E|$ represent the cardinal of graph elements in $G = (V, E)$ for all versions of MNP.

Proof. We have illustrated a construction of NMNP instance from an arbitrary MIS instance in the proof of Theorem 5. For any simple graph $G = (V, E)$ as an instance of MIS, one can construct corresponding instances for all other versions of MNP respectively.

Next, Figure 4(a) to Figure 4(e) provide feasible constructions of MNP instance form MIS instance $I : G$, according to different type of versions. To brief the description, let $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ and define $[k] = \{1, 2, \ldots, \}$, $n = |V|$, $m = |E|$, $d_i = d_G(v_i)$, $\forall i = 1, 2, \ldots, n$.

- NMNP: $G' = (V', E')$ with color mapping $c_V : V' \setminus \{s, t\} \to 2^{[m]}, v_i \mapsto \{k | v_i \text{ is incident to } e_k \ , k \in [m]\}$.
- NMMNP: $G' = (V', E')$ with color mapping $c_V : V' \setminus \{s, t\} \to [m], v_{i,j} \mapsto \sigma(v_{i,j})$, where $\sigma_i := \sigma\big|_{\{v_{i,1}, v_{i,2}, \ldots, v_{i,d_i}\}}$ satisfies $\sigma_i : \{v_{i,1}, v_{i,2}, \ldots, v_{i,d_i}\} \to \{k | v_i \text{ is incident to } e_k \ , k \in [m]\}, v_{i,j} \mapsto \sigma(v_{i,j})$ is a bijection.
- EMNP: $G' = (V', E')$ with color mapping $c_E : E' \to 2^{[m]}, e_i \mapsto \{k | v_i \text{ is incident to } e_k \ , k \in [m]\}$.
- EMMNP: $G' = (V', E')$ with color mapping $c_E : E' \to [m], e_{i,j} \mapsto \tau(e_{i,j})$, where $\tau_i := \tau\big|_{\{e_{i,1}, e_{i,2}, \ldots, e_{i,d_i}\}}$ satisfies $\tau_i : \{e_{i,1}, e_{i,2}, \ldots, e_{i,d_i}\} \to \{k | v_i \text{ is incident to } e_k \ , k \in [m]\}, e_{i,j} \mapsto \tau(e_{i,j})$ is a bijection.
- MMNP: $G' = (V', E')$ with color mapping $c : (V' \setminus \{s, t\}) \cup E' \to [m], x_{i,j} \mapsto \mu(x_{i,j})$, where $x_{i,j}$ denotes $v_{i,j}$ or $e_{i,j}$ for all $i \in [n]$, $j \in [d_i]$ and $\mu_i := \mu\big|_{\{e_{i,1}, v_{i,2}, \ldots, v_{i,2\lfloor d_n/2 \rfloor}, e_{i,d_i}\}}$ satisfies $\mu_i : \{e_{i,1}, v_{i,2}, \ldots, v_{i,2\lfloor d_n/2 \rfloor}, e_{i,d_i}\} \to \{k | v_i \text{ is incident to } e_k \ , k \in [m]\}, e_{i,j} \mapsto \mu(e_{i,j})$ is a surjection.

Note that there is a possibility that d_i is even for some $i \in [n]$, which leads to $2\lfloor d_i/2 \rfloor = d_i$. So that

$$\left|\{e_{i,1}, v_{i,2}, \ldots, v_{i,2\lfloor d_n/2 \rfloor}, e_{i,d_i}\}\right| = d_i + 1 > \left|\{k | v_i \text{ is incident to } e_k \ , k \in [m]\}\right| = d$$

and the mapping μ_i is just a surjection but not a bijection.

– MNP: $G' = (V', E')$

Actually, this is a universal case that all other versions are particular situation under some special restriction. Hence all above constructions between Figure 4(a) and Figure 4(e) can be regarded as a construction of MNP. So MNP can never have a better performance than any other versions of MNP.

We have shown kinds of polynomial-time constructions of versions of MNP instance $I' : (G', c)$ from arbitrary MIS instance $I : G$. According to this transformation, all 6 versions of MNP are hard to be approximated in different degree, based on the fact that MIS is hard to be approximated within $n^{1-\epsilon}$, where n represent the order of the graph.

Comparing the input size of created MNP instance $I' : (G', c)$ with original MIS instance $I : G$, one can easily find the results shown in Table 4. As it is proved in the proof of Theorem 5, any feasible solution of G

Table 4. Size and Order Relationship between MNP Instances $I' : (G', c)$ and MIS Instance $I : G$

	General	Mono-coloring
Mixed	—	$\lvert V'\rvert + \lvert E'\rvert \geq m + 2$ $\lvert V'\rvert + \lvert E'\rvert \leq m + n + 2$
Node	$\lvert V'\rvert = n + 2$	$\lvert V'\rvert = 2m + 2$
Edge	$\lvert E'\rvert = n$	$\lvert E'\rvert = 2m$

for MIS corresponds to a feasible solution of G' with color mapping c' for MNP with same cardinality, if we donate the MIS instance and MNP instance by $I : G$ and $I' : (G', c)$ respectively. Thus, the optimum value of $I : G$ for MIS equals to the optimum value of $I' : (G', c)$ for MNP. According to Theorem 4, note that

$$\rho = \min_{I \in \text{MIS}} \left\{\frac{\text{OPT}_I}{\text{SOL}_I}\right\} > n^{1-\epsilon}, \qquad \forall \epsilon > 0$$

for arbitrary polynomial-time algorithm $\mathcal{A}$ of MIS, unless $\mathcal{P} = \mathcal{NP}$, where I travels among all instances of MIS, and ρ, OPT_I, SOL_I represents performance ratio, optimum value and cardinality of the feasible solution given by algorithm $\mathcal{A}$ of MIS respectively. So that $\rho' = \min_{I' \in \text{MIS}} \left\{\frac{\text{OPT}'_I}{\text{SOL}'_I}\right\} = \min_{I \in \text{MIS}} \left\{\frac{\text{OPT}_I}{\text{SOL}_I}\right\} > n^{1-\epsilon}, \qquad \forall \epsilon > 0$, where ρ', OPT'_I, SOL'_I represents performance ratio, optimum value and cardinality of the feasible solution given by arbitrary polynomial-time algorithm $\mathcal{A}'$ of MNP respectively. Take NMNP as an example. The performance ratio of any NMNP algorithm satisfies that

$$\rho' > n^{1-\epsilon} = (n' - 2)^{1-\epsilon}, \qquad \forall \epsilon > 0.$$

Thus, NMNP is hard to be approximated within $n^{1-\epsilon}$ for graph order n, since $\lim_{n' \to +\infty} \frac{(n'-2)^{1-\epsilon}}{n'^{\ 1-\epsilon}} = 1$. Considering $2m = \sum_{v \in V} d_G(v) \leq n^2$, any polynomial-time MNP algorithm is hard to be approximated within performance ratio listed in Table 3 for versions of MNP, unless $\mathcal{P} = \mathcal{NP}$.

As we known, a optimization problem is $\mathcal{NP}$-hard is equivalent to its corresponding decision problem is $\mathcal{NP}$-complete. Therefore, we can confirm that the decision problem of MNP is $\mathcal{NP}$-complete. In fact, not only the corresponding decision problem is $\mathcal{NP}$-complete, but it is also a $\mathcal{NP}$-hard problem even if we ask the positive integer of the decision problem exactly equals to 2. Following discussion tells more about the fact.

Problem 7 (Decision Problem of MNP, DMNP) *Given a connected graph $G = (V, E)$ with two specified nodes s and t, as well as a positive integer $l \in \mathbb{N}_+$. Let $c : (V \setminus \{s, t\}) \cup E \to 2^{\mathrm{COLOR}}$ be a color mapping from graph G to a given color set* COLOR*. Decide whether there exists at least l color non-disrupting paths from s to t.*

Problem 8 (Strong Decision Problem of MNP, SDMNP) *Given a connected graph $G = (V, E)$ with two specified nodes s and t. Let $c : (V \setminus \{s, t\}) \cup E \to 2^{\mathrm{COLOR}}$ be a color mapping from the graph G to a given color set* COLOR*. Decide whether there exists color non-disrupting paths from s to t.*

Actually, strong decision problem of MNP is the special case for decision problem of MNP when l is exact equals to 2.

Theorem 7 ($\mathcal{NP}$-hard for SDMNP) *Strong decision problem of MNP is $\mathcal{NP}$-hard, unless $\mathcal{P} = \mathcal{NP}$.*

We will prove the theorem by reducing the set splitting problem, a well-known $\mathcal{NP}$-complete problem, to SDMNP in polynomial time. Before the proof, we need some preliminaries.

Definition 5 (Set Splitting Problem[29]) *In computational complexity theory, the set splitting problem is the following decision problem: given a family $\mathcal{F}$ of subsets of a finite set S, decide whether there exists a partition of S into two subsets S_1, S_2 such that all elements of $\mathcal{F}$ are split by this partition, i.e., none of the elements of $\mathcal{F}$ is completely in S_1 or S_2.*

Set splitting is one of Garey&Johnson's classical $\mathcal{NP}$-complete problems[29]. In the following, we will prove the $\mathcal{NP}$-hardness of SDMNP by polynomial reduction from set splitting problem.

Proof. Given an arbitrary set splitting instance, we will transform it into a SDMNP instance in polynomial time.

Assume the set splitting instance I consists of a finite set $S = \{s_1, s_2, \ldots, s_k\}$ and a family of subsets $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ for certain positive integers k and n. To simplify the notation, let $S_i = \{s_{i_1}, s_{i_2}, \ldots, s_{i_{L_i}}\}$ for all $i \in [n]$, where $L_i = |S_i|$. Then we can construct the instance I' including a simple connected graph $G = (V \cup \{s, t\}, E)$ with a color mapping $c : V \to S$ for SDMNP, corresponding to set splitting problem instance I as follows.

- Create vertex set $V_i := \{v_{i,1}, v_{i,2}, \ldots, v_{i,L_i}\}$ for subset $S_i \in \mathcal{F}$, for all $i \in [n]$.
- Let $V := V_1 \cup V_2 \cup \cdots \cup V_n = \{v_{i,j} \big| s_{i_j} \in S_i, \forall i \in [n], j \in [L_i]\}$, where $L_i = |S_i|$.
- Let $E := \{uw \big| u \in V_i \text{ and } w \in V_{i+1}, \forall i = 0, 1, \ldots, n\}$, with the setting $V_0 = \{s\}$ and $V_{n+1} = \{t\}$.
- Let color mapping $c : V \to S, v_{i,j} \mapsto s_{i_j}, \forall i \in [n], j \in [L_i]$.

Figure 5 illustrates the graph of the instance I'.

It is obvious that there exists non-disrupting paths in graph $G = (V \cup \{s, t\}, E)$ with color mapping $c : V \to S$ if family of subsets $\mathcal{F}$ can be split. In the other hand, non-disrupting paths of graph G naturally generate two non-intersection subsets of S, and, further more, a partition of set S is formed at once.

Therefore, the SDMNP is $\mathcal{NP}$-hard unless one can solve set splitting problem, a common known $\mathcal{NP}$-complete problem, in polynomial time.

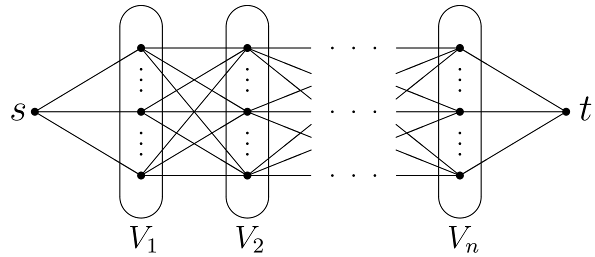


Fig. 5. Graph of Strong Decision Problem of MNP Instance

6 Concluding Remarks

This paper introduces six new multi-path computation problems with general failure interdependency models in complex networks such as smart grid communication networks. We showed that the problems are generally NP-hard and are hard to approximate. We believe this paper will serve as one of seed efforts to provide multi-path routing algorithms for various complex network with unique failure interdependency models. As a future work, we plan to introduce approximation algorithm for the problems as well as heuristic algorithm with superior average performance.

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